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A SELFSIMILAR PROBLEM ON THE ACTION OF A SUDDEN LOAD ON THE BOUNDARY OF AN ELASTIC HALF-SPACE*

A.G. KULIKOVSKII and E.I. SVESHNIKOVA

The solution of the non-linear problem of the action of a constant stress suddenly applied to the plane boundary of an elastic half-space that has homogeneous prestrain is investigated. The problem is selfsimilar, and its solution is constructed from shock and selfsimilar simple waves investigated earlier /1-5/. The problem under consideration is the necessary element that should be contained in solutions of different non-stationary problems, for instance, in the problem of the decay of an arbitrary initial discontinuity. Moreover, the selfsimilar solution constructed below represents the asymptotic form long times of the corresponding non-selfsimilar problems when the stress on the half-space boundary varies from some values to others according to an arbitrary law over a limited time.

1. Formulation of the problem. A homogeneous isotropic non-linearly elastic medium is given by its internal energy $U(\varepsilon_{ij}, S)$ in the form /1-5/

$$\Phi = \rho_0 U = {}^1/_2 \lambda I_1^2 - \mu I_2 - \beta I_1 I_2 + \gamma I_3 - \delta I_1^3 - \xi I_2^2 + \rho_0 T_0 (S - S_0) + \text{const} \tag{1.1}$$

$$I_1 = \varepsilon_{ii},\ I_2 = \varepsilon_{ij}\varepsilon_{ij},\ I_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial \xi_j} - \frac{\partial w_j}{\partial \xi_i} + \frac{\partial w_k}{\partial \xi_i}\frac{\partial w_k}{\partial \xi_j}\right)$$

Here S is the entropy, ε_{ij} are the components of Green's strain tensor, w_i is the displacement vector, ρ_0 is the density in the unstressed state, and ξ_i are the Lagrange coordinates that are rectangular Cartesian coordinates in the unstressed state.

The medium that possesses a small homogeneous initial strain occupies the half-space $\xi_3 \geqslant 0$. At the time $t = 0$ a stress that alters the state of strain on the boundary is applied to the boundary $\xi_3 = 0$ and later remains constant. The problem is selfsimilar, and the solution depends on ξ_3, t. A perturbation from the boundary in the domain $\xi_3 > 0$ propagates in the form of plane strain waves in which only the following components of the displacement gradient vary: $\partial w_1/\partial \xi_3 = u$, $\partial w_2/\partial \xi_3 = v$, $\partial w_3/\partial \xi_3 = w$. We designate by U, V, w^c, respectively, the initial magnitudes of these strain components, and we denote those values which they acquire on the boundary subjected to the action of the suddenly applied stress by u_*, v_*, w_*, respectively.

In addition to the above, the medium also possesses other strain components that do not vary in this problem and play the part of parameters. These components are ε_{11} and ε_{22}. The

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quantity ε_{12} can be considered to be zero if the directions of the axes ξ_1 and ξ_2 are selected appropriately.

We will assume the initial and boundary strains to be small, of the same order of smallness, which we will agree to denote by ε. The expansion (1.1) of the function Φ in the small strains of $\sim\varepsilon$ yields a representation of the potential for an arbitrary isotropic elastic medium with accuracy to ε^4 /1–3/.

One quasilongitudinal and two quasitransverse simple or shock waves can propagate in the direction of the positive part of the ξ_3 axis. They will also be used to construct the solution.

2. Utilization of quasilongitudinal waves in the solution. The velocities of the quasilongitudinal waves, both the simple and the shock, exceed the velocities of any quasitransverse waves by the finite quantity $\lambda + \mu$. The component w undergoes the main change in these waves. The change in u and v is small, of the second order in ε or even less.

Therefore, the state of strain behind the quasilongitudinal waves is determined by the components ε_{11}, ε_{22}, w_*, $U + O(\varepsilon^2)$, $V + O(\varepsilon^2)$. We take it as the new initial state over which the quasitransverse waves propagate, and for simplicity we denote the initial shear strains therein by U, V, as before. Now, the problem can be solved for some quasitransverse waves. The change that they introduce into the component w will be of the second order of smallness in ε. If this correction must be taken into account, the solution of the problem can be continued by successive approximations.

3. Solution with quasitransverse waves. The component w can be eliminated from investigations /1, 3/ (expressed in terms of u and v) in quasitransverse waves. Further construction of the solution can be carried out in the uv plane where the point U, V portraying the initial state, and the point u_*, v_* (the final state) must be connected continuously by using the integral curves of the quasitransverse simple waves /2/ and the evolutionary sections of the shock adiabat /3/. It is here necessary to conserve the sequence of the wave succession as a function of the velocity.

The integral curves of simple waves are described by the differential equations

$$\frac{dv}{du} = \frac{v^2 - u^2 - G \pm [(v^2 - u^2 - G)^2 + 4u^2v^2]^{1/2}}{2uv} \tag{3.1}$$

$$G = (2\mu - {}^3/_2\gamma)(\varepsilon_{22} - \varepsilon_{11})/\varkappa$$

$$\varkappa = \mu - (\mu - \beta - {}^3/_2\gamma)^2/(\lambda + \mu) - 2\xi .$$

The form of these lines and the change in the characteristic velocities along them are investigated in /2/. Since simple waves have a tendency to breaking for certain directions of the change of parameters therein, the solution of the selfsimilar problem can then consist of just not-breaking simple and shock waves.

The shock adiabat of quasitransverse shock waves is given in the uv plane by the equation

$$(u^2 + v^2 - R^2)(Uv - Vu) - 2G(u - U)(v - V) = 0 \tag{3.2}$$

(here $R^2 = U^2 + V^2$). The shock adiabat and the segments extracted thereon that simultaneously satisfy the conditions of a non-decrease in entropy and evolutionarity /1, 3/ are shown in Fig.1 (the heavy continuous line for media with $\varkappa > 0$, and the heavy dashed line for media with $\varkappa < 0$).

The initial point $A(U, V)$ can always be located in the first quadrant of the uv plane. An arbitrary finite (boundary) state u_*v_* can be portrayed by any point in this plane. Because of the anisotropy of the initial state of strain, quasitransverse waves, both simple and shock, are separated into fast waves, for which the velocity of the jump W satisfies the conditions

$$c_2^- \leqslant W \leqslant c_2^+,\ c_1^+ \leqslant W$$

and slow waves for which

$$c_1^- \leqslant W \leqslant c_1^+,\ 0 \leqslant W \leqslant c_2^- .$$

Here c_i^+ and c_i^- are the characteristic velocities behind and ahead of the discontinuity, respectively. In the case when the evolutionary segment of the shock adiabat adjoins the initial point $A(U, V)$, we call the corresponding shocks waves of the first kind, and in the remaining cases waves of the second kind.

We denote the slow and fast simple waves by R_1 and R_2, respectively, and the slow and fast shocks of the first kind by S_1, S_2 and the slow and fast shocks of the second kind by S_1^* and S_2^*.

When it helps the comprehension, we shall indicate the initial and final point of the velocity of the jump W in the uv plane.

The Jouguet /1, 3/ shocks for which the velocity of the wave W agrees with one of the characteristic velocities $c_i^{\pm}$, i.e., the equality signs are satisfied under evolutionarity conditions, are of special value when constructing the solution. The shock velocity is $W = c_2^-$ at the points K, F, K', F', D in Fig.1, and at the other Jouguet points $W_L = c_1^-$, $W_E = c_1^+$, $W_H = c_1^+$, $W_J = c_2^+$. If $W = c^+$, then a simple wave of the same type can follow behind the Jouguet shock. If $W = c^-$, then the Jouguet shock can follow directly behind a simple wave of the same type.

The form of the shock adiabat as well as the number and location of the evolutionary segments thereon depend on the sign of the function D for $\varkappa > 0$ and of D_1 for $\varkappa < 0$ /3/

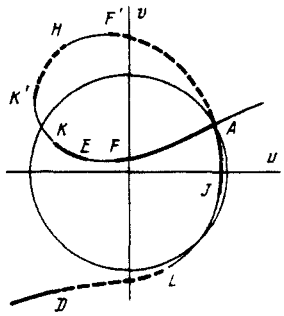

Fig.1

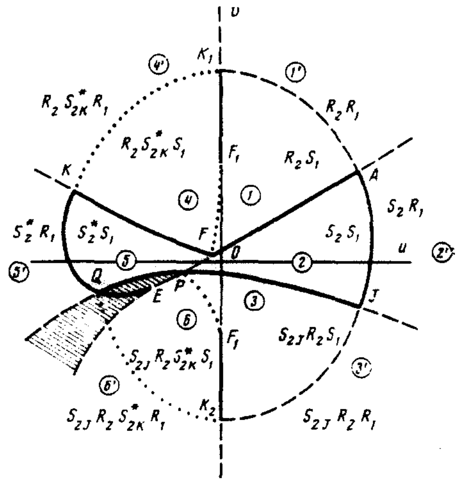

Fig.2

$$D(U, V, G) \equiv c_2^- - c_1^+(E), \quad D_1(U, V, G) \equiv c_2^- - c_1^+(H). \tag{3.3}$$

The equations $D = 0$ and $D_1 = 0$ are shown by closed lines containing the origin and having the dimensions $\sim \sqrt{G}$ /3/ in the plane of the initial strains UV. Within the appropriate curves D and D_1 are positive; outside they are negative. The shock adiabat is displayed in Fig.1 for the maximum number of evolutionarity sections, which corresponds to $D < 0$ (for $\varkappa > 0$) and $D_1 > 0$ (for $\varkappa < 0$). When $D = 0$ the points K, E, F merge into one point E, when $D_1 = 0$, the points K', H, F' merge into the point H.

The construction of the solution in each such case should be examined separately.

4. The case when $\varkappa > 0, G/R^2 \ll 1$. The initial point A lies in the first quadrant of the uv plane outside the curve $D = 0$, i.e., in the domain $D < 0$. In this case, two fast shocks S_2 and S_2^* and one slow shock S_1 exist. Let R^2 first be so much greater than G that the whole evolutionary section AJ of the shock adiabat also lies outside the curve $D = 0$. The solution is constructed separately in each of the domains 1—6, 1′ — 6′ displayed in Fig.2. We comment on the construction of these solutions.

The evolutionary sections AF, EK, AJ of the shock adiabat of the initial point A are shown by heavy lines in Fig.2 (here and in subsequent figures). The non-evolutionary section of the same shock adiabat is portrayed by the fine line FPE. It is obviously possible to go from the point A of the slow shock S_1 to any point of the segment AF. In order to be incident in domain 1 above AF, it is first necessary to go along the segment of the integral curve AK_1 of the fast simple wave R_2. A continuous selfsimilar solution exists here just up to the vertical axis. The sections of the simple wave integral curves used in Fig.2 and all subsequent figures are shown by dashes. From any point of the arc AK_1 we go to the domain 1′ by the slow simple wave R_1 and to the domain 1 by the slow shock S_1. The evolutionary sections of these waves S_1 terminate at points of the segment FF_1 in the second quadrant. At these points $W = c_2^-$. The point F_1 on the shock adiabat from the point K_1 has the coordinates $u = 0, v = 2G/R$ /4/. Thus, the solution has the form R_2S_1 in the domain 1 (AFF_1K_1) and R_2R_1 in domain 1′.

The fast shock S_2 with $W_1 < c_2(A)$ leads to points of the segment AJ. The slow simple wave R_1 starting behind it goes to any point of the domain 2′, and the solution in it has the form S_2R_1.

A slow wave S_1 with velocity $W_2 \leqslant W_1$ can go from any state on AJ after the first fast wave S_2 having the velocity W_1. The evolutionary segments of the slow waves S_1 go from the state AJ into the domain 2 and should terminate at point where $W_2 = c_2^-$ (since all the points AJ are in the domain $D < 0$ by our assumption). As shown in /5/, the adiabats of these slow waves certainly intersect the initial adiabat on the section FP, i.e., fill the whole domain 2. At the intersections $W_2 = W_1$. It is impossible to use segments of the adiabat S_1 behind the intersection with FP for the solution since there would be $W_2 > W_1$ and the second wave would overtake the first. Therefore, the sequence S_2S_1 yields a solution in domain $2\,(APJ)$. The evolutionary part of the shock adiabat S_1, constructed for the point J, is the lower boundary of this domain PJ, where the point P is the end of this evolutionary segment since $W_2 = W_{AJ} = c_2^-(J)$ there. For small G/R^2 it lies between the points E and F /5/.

To construct the solution in domains $3 - 3'$, the Jouguet shock S_{2J} is used at the point J. A fast simple wave can follow behind it. It is possible to go to the vertical axis along its integral curve JK_2. Further construction of the solution is the same as for domains $1 - 1'$. The solution obtained has the following form: in domain 3 $(JPF_2K_2) - S_{2J}R_2S_1$ and in domain $3' - S_{2J}R_2R_1$.

We use a fast shock of the second type S_2^* to transfer into the left half-plane of the uv plane. One such wave corresponds to a jump from the state A at the points of the segment KE of the initial adiabat, for it $W_1 \geqslant c_2(A)$. A slow simple wave R_1 in the state of the domain $5'$ can go along each of the states obtained. The solution for any point of domain $5'$ has the form $S_2^*R_1$. The integral curve of the simple wave R_1 touching the shock adiabat at the point E $(W_{AE} = c_1^+)$/3/ is the lower boundary of this domain.

A slow shock S_1 can go over each of the states of KE. The evolutionary sections of the slow wave shock adiabats intersect the initial shock adiabat in the section EF /5/ and therefore cover the whole domain 5. At the intersections $W_1 = W_2 = c_2^-$. But $c_2^- \geqslant c_2(A)$. Consequently, the parts of these evolutionary segments of S_1 after the intersection with EF are unsuitable for the solution since we would have $W_2 > W_1$ on them. Therefore, the solution in domain 5 consists of a sequence of two jumps $S_2^*S_1$.

The solution is constructed as follows for the domains $4 - 4'$. A fast simple wave R_2 from the initial state A travels to any point of the arc AK_1. Behind it travels a Jouguet shock of the second kind S_{2K}^* with velocity $W = c_2^-$. It is shown in /4/ that the closer the initial point is to the vertical axis, the closer is the point K to the vertical axis on its side. Consequently, by selecting the state on the arc AK_1 in an appropriate manner, a wave S_{2K}^* can arrive at any point of the arc KK_1. The slow simple wave R_1 travelling behind it completes the solution for the domain $4': R_2S_{2K}^*R_1$. If a slow shock travels behind S_{2K}^*, then its evolutionary sections will go into domain 4. We have $W = c_2^-$ at the points KK_1. The evolutionary sections of S_1 can be used for the solution only to states where $W_2 = W_1$ and since $W_1 = c_2^-$, this is the point of the segment FF_1, the boundary of domain 1. Thus the solution $R_2S_{2K}^*S_1$ is found in the whole domain 4.

The solution in the domain 6 (QPF_2K_2) $S_{2J}R_2S_{2K}^*S_1$ is similarly found, and in domain $6'$ also: $S_{2J}R_2S_{2K}^*R_1$.

The point Q through which the upper boundary of domain 6 passes is a Jouguet point of the shock adiabat constructed for the point J, and therefore, lies in the third quadrant of the uv plane. It is the state behind the fast S_{2K}^* wave travelling behind the other fast wave S_{2J}. Two fast waves can follow each other only at the identical velocity $W_1 = W_2 = c_2(J)$. This combination of two waves can be considered as one composite jump since the states before the first and after the second wave satisfy the conservation laws with the same constants as for the first wave. Hence, the state behind the second wave (the point Q) also lies on the shock adiabat of the initial point A. The velocity W varies along this (initial) adiabat so that $W_P = W_Q = c_2(J)$, while it reaches a maximum at the point E. It is hence clear that the point Q always lies on the segment KE /5/. This results in the upper boundary of the domains 6 and $6'$ passing within the domains 5 and $5'$. The domains 5 and $6, 5'$ and $6'$ have an intersection and the solution in the shaded zone in Fig. 2 is not unique. The solution on the segment PF will be ambiguous as is seen from the construction of the solution in domains 2 and 5.

As $G/R^2 \to 0$ the points P and F approach the origin, all the boundaries of the domains approach circles and rays, while the domain of ambiguity is converted into an angular sector with vertex at the point O.

5. The case $\varkappa > 0$, $D < 0$. We will now enlarge G/R^2 while conserving the requirement $D(A) < 0$ so that the initial point A remains outside the curve $D = 0$. From a certain time of the change in G/R^2 a part of the evolutionary segment AJ on the initial shock adiabat falls inside the domain $D \geqslant 0$.

In conformity with (3.3), we should have $W_2 = c_1^+ = c_1^-$ at the end of the evolutionary section for the shock adiabat of the slow wave travelling from the intersection of the initial shock adiabat with the line $D = 0$. If $W = W_{AJ}$, then the condition $c_1^+ = c_2^- = c_2(J)$ means that the points P and Q in Fig. 2 merge at the point of tangency E of the shock adiabats of the first and second waves. All the shock adiabats, starting at points of the segment AJ intersect the initial shock adiabat at two points on different sides of the point E /5/. Consequently, only the point E where the Jouguet condition $W_{AE} = W_{AJ} = W_{JE} = c_1(E)$ is satisfied on both and they are also tangent to the integral curve R_1, can be the point of tangency of these adiabats. Taking account of the monotonicity of the changes in W in the segment AJ, we hence conclude that the segment AJ intersects the curve $D = 0$ at just one point. The point J is therefore the first of the states of the segment AJ to fall into the domain $D \geqslant 0$.

Now, let the point J lie within $D > 0$. The intersection of the initial shock adiabat with the line $D = 0$ occurs at another point M_1 of the segment AJ (Fig.3). For a slow wave travelling from this point $c_2^- = c_1^- = W_{AM_1} < W_{AJ}$ and its shock adiabat is now tangent to the initial adiabat at the point E. But the evolutionary segments of the shock adiabats from the state M_1J (within $D > 0$) are terminated by the points EE' of the integral curves R_1. Therefore, the domain 2 with solution S_2S_1 is bounded by the line $AFEE'J$.

The solutions in all the domains $1-6, 1'-6'$ are constructed exactly as in Sec.4 and have the same form as in the domains with the same numbers in Fig.2. The point E_2 is obtained as the end of the evolutionary section of the shock adiabat S_1 travelling from the point M_2, the points of intersection of the fast simple wave integral curve departing from the point J and the line $D = 0$. Unlike the preceding, two new domains 7 and 8 appear. The right boundaries EE_2 thereon can drop by using the shocks S_1 with velocities $W = c_1^-$. Consequently, to the left of the line EE_2 the solution can be continued along integral curves of simple waves R_1 which are tangent to the corresponding shock adiabats at the points EE_2. The solution in domain 8 will be $S_2S_{1E}R_1$, and $S_{2J}R_2S_{1E}R_1$ in domain 7.

6. The case $\varkappa > 0$, $D > 0$. The initial point A now lies in the domain $D > 0$. The whole section AJ of the shock adiabat and the segments M_1A and JM_2 of the integral curves of the family R_2 that are its continuation is there (Fig.4) The arcs K_1M_1 and K_2M_2 of the integral curves lie in the domain $D \leqslant 0$. The points E_1 and E_2 are the ends of the evolutionary sections of slow waves travelling from the boundary points M_1 and M_2, respectively. The jump velocity is $W = c_1^-$ at the points E_1 and E_2 and simple waves R_1 can follow directly behind such Jouguet shocks. Their integral curves are, respectively, the upper boundary of the domain 9 and the lower for domain 7. The solution in domain 9 has the form $R_2S_{1E}R_1$. In the remaining domains the structure of the solution is the same as in Figs.1 and 2 in the domains with the same numbers. As the quantity R^2/G diminishes further, the domains 4, 4', 6, 6' depart for infinity. In the limit when $R^2 \ll G$ and $u_*^2 + v_*^2 \ll G$, the integral curves of the simple waves and the part of the shock adiabats used in the solution are close to lines parallel to the u and v axes. This solution was presented in /6/.

7. The case $\varkappa < 0$. For media with $\varkappa < 0$ the procedure for constructing the solution is analogous to the preceding, except that in place of the function $D(U, V, R)$ the function $D_1(U, V, R)$ plays the same part. Becuase large distinct shocks exist for $\varkappa < 0$ (see Fig.1), a larger number of domains with a different kind of structure of the solution is obtained when constructing the solution of the selfsimilar problem under investigation. For the case when the initial point A is in the domain $D_1 > 0$, the pattern of the solution is displayed in Fig.5. The intersection of the integral curve (or shock adiabat) with the line $D_1 = 0$ is

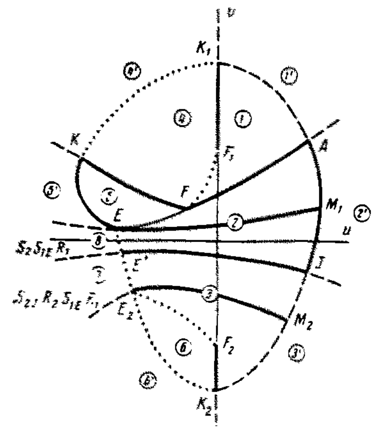

Fig.3

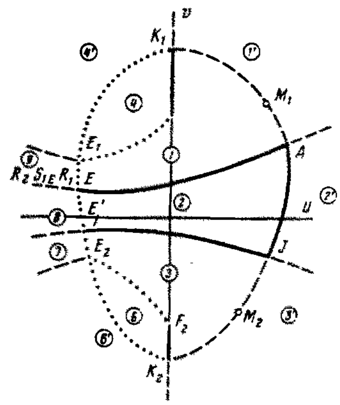

Fig.4

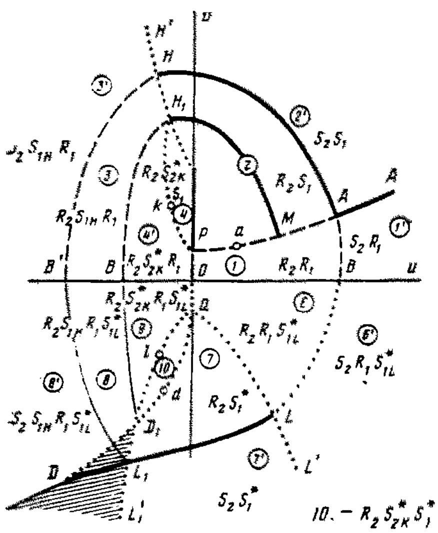

Fig. 5

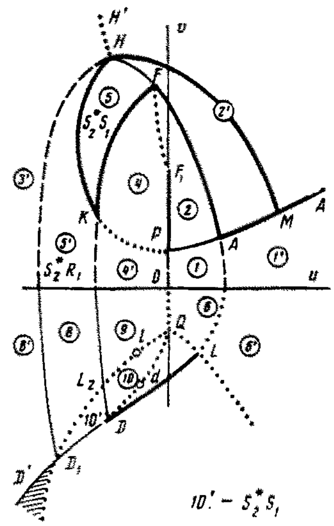

Fig.6

denoted by the point M. In the shaded domain in Fig.5, the solution turns out to be ambiguous. As $G/R^2 \to 0$ the boundaries of all the domains tend to circles and rays, the points P, Q, H_1, D_1 tend to the origin, and the sector of ambiguity of the solution $D'D_1L_1'$ has a vertex at the point O.

When the point A mapping the initial state lies in the domain $D_1 < 0$, zones with a different kind of solution are presented in Fig.6. The structure of the solution in each of the domains is the same as in the zones with the same numbers in Fig.5. New domains 5 and 5′ appear in place of 3, and also a domain 10′.

As $G \to \infty$, all the integral curves and sections of the shock adiabats used in the solution in the finite part of the uv plane approach lines parallel to the coordinate axes. The point D_1 is the vertex of the non-uniqueness sector, and goes to infinity.

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ON AN EFFECTIVE ALGORITHM FOR MINIMIZING GENERALIZED TREFFTZ FUNCTIONALS OF LINEAR ELASTICITY THEORY*

V. YA. TERESHCHENKO

The problem of minimizing the generalized Trefftz functionals of three-dimensional elasticity theory results in a minimax problem for the Lagrangian. An algorithm is proposed for searching for the saddle point in coordinate functions not subjected to any constraints in the domain and on the boundary (this is the efficiency of the algorithm). The convergence of the approximate solution is investigated.

The Trefftz variational method /1/ is convenient for solving boundary value problems of mathematical physics in that the dimensionality of the problem being solved is reduced because of its reduction to the solution of equations defined on the domain boundary. At the same time, when constructing the solution using the Ritz process, say, the coordinate functions should be selected so that they satisfy the differential equation of the boundary value problem in the domain, which is a serious constraint. An approach is proposed below that uses Lagrange multipliers to reduce this constraint when minimizing the generalized Trefftz functionals of the fundamental boundary value problems of linear elasticity theory. The results obtained can also be used to minimize the classical Trefftz functionals of the boundary value problems of mathematical physics /1/.

Generalized Trefftz functionals were constructed in /2, 3/ for the fundamental problems of linear elasticity theory with continuous and discontinuous elasticity coefficients. The functionals are minimized in solutions (ordinary or generalized) for the linear equilibrium equation for an elastic medium in displacements. Assuming the existence of a coordinate system of functions satisfying the equilibrium equation (in the generalized sense) in /4/, the Ritz process was investigated for solving problems to minimize the generalized Trefftz functionals in an example of the second boundary value problem of three-dimensional elasticity theory. The practical construction of the above-mentioned coordinate system is a fairly complex problem. At the same time, the differential equation of the boundary value problem in whose solutions the minimum of the functionals is sought, can be considered as a linear constraint in the problem of minimizing the Trefftz functionals. Then such a minimization problem with linear constraints can be reduced to the minimax problem of a certain Lagrangian (by using reciprocity theory).

1. The notation in /2,3/ is used henceforth. Let $\Phi(u)$ be a generalized Trefftz functional of one of the fundamental boundary value problems of linear elasticity theory with the domain of definition

$$D_1(\Phi) = \{u \in W_2^2(G) \mid Au \in L_2(G), Au = K\}$$

which can be extended as follows:

$$D_2(\Phi) = \left\{u \in W_2^1(G) \,\Big|\, 2\int_G W^{\cdot}(u, v)\, dG - \int_S t(u)v\, ds = \int_G Kv\, dG, \quad \forall v \in W_2^1(G)\right\}.$$

Here $u \in D_2(\Phi)$ is the generalized solution of the equilibrium equation $Au = K$ in the

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